

# APPROXIMATION BY A GENERALIZATION OF THE JAKIMOVSKI -LEVIATAN OPERATORS

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**ABSTRACT.** In this paper, we introduce a Kantorovich type generalization of Jakimovski-Leviatan operators constructed by A. Jakimovski and D. Leviatan (1969) and the theorems on convergence and the degree of convergence are established. Furthermore, we study the convergence of these operators in a weighted space of functions on a positive semi-axis.

## 1. INTRODUCTION

In approximation theory, Szász type operators and Chlodowsky type generalizations of these operators have been studied intensively [see [1], [2], [15], [12], [10], [16], [17], [11], [8] and many others]. Also orthogonal polynomials are important area of mathematical analysis, mathematical and theoretical physics. In mathematical analysis and in the positive approximation processes, the notion of orthogonal polynomials seldomly appears. Cheney and Sharma [7] established an operator

$$P_n(f; x) = (1-x)^{n+1} \exp\left(\frac{tx}{1-x}\right) \sum_{k=0}^{\infty} f\left(\frac{k}{k+n}\right) L_k^{(n)}(t) x^k \quad (1.1)$$

where  $t \leq 0$  and  $L_k^{(n)}$  denotes the Laguerre polynomials. For the special case  $t = 0$ , the operators given by (1.2) reduce to the well-known Meyer-König and Zeller operators [14].

In view of the relation between orthogonal polynomials and positive linear operators have been investigated by many researchers (see [17],[8]).

One of them is Jakimovski and Leviatan 's study. In 1969, authors introduced Favard-Szász type operators  $P_n$ , by using Appell polynomials are given with

$$g(u) = \sum_{n=0}^{\infty} a_n u^n, g(1) \neq 1 \text{ be an analytic function in the disk } |u| < r \text{ (} r > 1 \text{) and}$$

$$p_k(x) = \sum_{i=0}^k a_i \frac{x^{k-i}}{(k-i)!}, (k \in \mathbb{N}) \text{ be the Appell polynomials defined by the identity}$$

$$g(u)e^{ux} \equiv \sum_{k=0}^{\infty} p_k(x) u^k. \quad (1.2)$$

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Let  $E[0, \infty)$  denote the space of exponential type functions on  $[0, \infty)$  which satisfy the property  $|f(x)| \leq \beta e^{\alpha x}$  for some finite constants  $\alpha, \beta > 0$ .

In [1], the authors considered the operator  $P_n$ , with

$$P_n(f; x) = \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k(nx) f\left(\frac{k}{n}\right) \quad (1.3)$$

for  $f \in E[0, \infty)$  and studied approximation properties of these operators, as well as the analogue to Szász's results.

If  $g(u) \equiv 1$ , from (1.2) we obtain  $p_k(x) = \frac{x^k}{k!}$  and we obtain classical Szász-Mirakjan operator which is given by

$$S_n(f; x) = e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right).$$

In 1969, Wood [6] showed that the operators  $P_n$  are positive if and only if  $\frac{a_k}{g(1)} \geq 0$ , ( $k = 0, 1, \dots$ ). In 1996, Ciupa [2] was studied the rate of convergence of these operators. In 1999, Abel and Ivan [18] showed an asymptotic expansion of the operators given by (1.3) and their derivatives. In 2003, İspir [15] showed that the approximation of continuous functions having polynomial growth at infinity by the operator in (1.3). In 2007, Ciupa [3] defined Modified Jakimovski-Leviatan operators and studied rate of convergence, order of approximation and Voronovskaya type theorem. Recently, Büyükyazıcı and et. al, [12] studied approximation properties of Chlodowsky type Jakimovski-Leviatan operators. They proved Voronovskaya-type theorem and studied the convergence of these operators in a weighted space by using a new type of weighted modulus of continuity.

In this paper, we consider the following Kantorovich type generalization of Jakimovski-Leviatan operators given by

$$L_n^*(f; x) = \frac{e^{-nx}}{g(1)} \frac{n}{b_n} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} f(t) dx \quad (1.4)$$

with  $b_n$  a positive increasing sequence with the properties

$$\lim_{n \rightarrow \infty} b_n = \infty, \quad \lim_{n \rightarrow \infty} \frac{b_n}{n} = 0 \quad (1.5)$$

and  $p_k$  are Appell polynomials defined by (1.2). Recently, some generalizations of (1.3) operators have been considered in [2], [3] and [9].

## 2. SOME APPROXIMATION PROPERTIES OF $L_n^*(f; x)$

In approximation theory, the positive approximation processes discovered by Korovkin play a central role and arise in a natural way in many problems connected with functional analysis, harmonic analysis, measure theory, partial differential equations and probability theory.

Let  $C_E[0, \infty)$  denote the set of all continuous functions  $f$  on  $[0, \infty)$  with the property that  $|f(x)| \leq \beta e^{\alpha x}$  for all  $x \geq 0$  and some positive finite  $\alpha$  and  $\beta$ . For a

fixed  $r \in \mathbb{N}$ , we denote by

$$C_E^r[0, \infty) = \left\{ f \in C_E[0, \infty) : f', f'', \dots, f^{(r)} \in C_E[0, \infty) \right\}.$$

**Lemma 1.** *The operators  $L_n^*(f; x)$  defined by (1.4) satisfy the following equalities.*

$$L_n^*(1; x) = 1, \quad (2.1)$$

$$L_n^*(t; x) = x + \frac{g'(1)}{g(1)} \frac{b_n}{n} + \frac{b_n}{n}, \quad (2.2)$$

$$L_n^*(t^2; x) = x^2 + \frac{b_n}{n} x \left( \frac{g(1) + 2g'(1)}{g(1)} + 1 \right) + \frac{b_n^2}{n^2} \left( \frac{2g'(1)}{g(1)} + \frac{g''(1)}{g(1)} + \frac{1}{3} \right). \quad (2.3)$$

**Lemma 2.** *The central moments of the operators  $L_n^*(f; x)$  are given by*

$$\begin{aligned} L_n^*(t - x; x) &= \frac{g'(1)}{g(1)} \frac{b_n}{n} + \frac{b_n}{n} \\ L_n^*((t - x)^2; x) &= \frac{b_n^2}{n^2} \left( \frac{2g'(1)}{g(1)} + \frac{g''(1)}{g(1)} + 1 \right) \end{aligned} \quad (2.4)$$

**Theorem 1.** *If  $f \in C_E[0, \infty)$ , then  $\lim L_n^*(f) = f$  uniformly on  $[0, a]$ .*

*Proof.* From (2.1)-(2.3), we have

$$\lim L_n^*(e_i; x) = e_i(x), \quad i \in \{0, 1, 2\},$$

where  $e_i(x) = t^i$ . Applying the Korovkin theorem [5], we obtain the desired result.  $\square$

In this section, we deal with the rate of convergence of the  $L_n^*(f; x)$  to  $f$  by means of a classical approach, the second modulus of continuity, and Peetre's K-functional.

Let  $f \in \tilde{C}[0, \infty)$ . If  $\delta > 0$ , the modulus of continuity of  $f$  is defined by

$$\omega(f, \delta) = \sup_{\substack{x, y \in [0, \infty) \\ |x - y| \leq \delta}} |f(x) - f(y)|,$$

where  $\tilde{C}[0, \infty)$  denotes the space of uniformly continuous functions on  $[0, \infty)$ . It is also well known that, for any  $\delta > 0$  and each  $x \in [0, \infty)$ ,

$$|f(x) - f(y)| \leq \omega(f, \delta) \left( \frac{|x - y|}{\delta} + 1 \right).$$

The next result gives the rate of convergence of the sequence  $L_n^*(f; x)$  to  $f$  by means of the modulus of continuity.

**Theorem 2.** *If  $f \in C_E[0, \infty)$ , then for any  $x \in [0, a]$  we have*

$$|L_n^*(f; x) - f(x)| \leq \left\{ 1 + \frac{1}{\delta} \left( \sqrt{\theta_n} \right) \right\} \omega(f, \delta).$$

where

$$\theta_n = \frac{b_n^2}{n^2} \left( \frac{2g'(1)}{g(1)} + \frac{g''(1)}{g(1)} + 1 \right).$$

*Proof.*

$$\begin{aligned}
|L_n^*(f; x) - f(x)| &\leq \frac{e^{-nx}}{g(1)} \frac{n}{b_n} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} |f(s) - f(x)| ds \\
&\leq \frac{e^{-nx}}{g(1)} \frac{n}{b_n} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} \omega(f, \delta) \left(\frac{|s-x|}{\delta} + 1\right) ds \\
&\leq \left\{ 1 + \frac{e^{-nx}}{g(1)} \frac{n}{b_n} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} |s-x| ds \right\} \omega(f, \delta)
\end{aligned}$$

By using the Cauchy-Schwarz inequality for integration, we get

$$\int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} |s-x| ds \leq \frac{1}{\sqrt{n}} \left( \int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} |s-x|^2 ds \right)^{1/2}$$

which holds that

$$\sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} |s-x| ds \leq \frac{1}{\sqrt{n}} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \left( \int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} |s-x|^2 ds \right)^{1/2}.$$

If we apply the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
|L_n^*(f; x) - f(x)| &\leq \left\{ 1 + \frac{1}{\delta} \left( \frac{e^{-nx}}{g(1)} \frac{n}{b_n} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} |s-x|^2 ds \right) \right\} \omega(f, \delta) \\
&= \left\{ 1 + \frac{1}{\delta} \left( \sqrt{L_n^*((s-x)^2; x)} \right) \right\} \omega(f, \delta) \\
&\leq \left\{ 1 + \frac{1}{\delta} \left( \sqrt{\theta_n} \right) \right\} \omega(f, \delta).
\end{aligned}$$

Now if we choose  $\delta = \sqrt{\theta_n}$ , it completes the proof.  $\square$

Now we remember the second modulus of continuity of  $f \in C_B[0, \infty)$  which is defined by

$$\omega_2(f; \delta) = \sup_{0 < t < \delta} \|f(\cdot + 2t) - 2f(\cdot + t) + f(\cdot)\|_{C'_B}$$

where  $C_B[0, \infty)$  is the class of real valued functions defined on  $[0, \infty)$  which are bounded and uniformly continuous with the norm  $\|f\|_{C_B} = \sup_{x \in [0, \infty)} |f(x)|$ .

Peetre's  $K$ -functional of the function  $f \in C_B[0, \infty)$  is defined by

$$K(f; \delta) = \inf \left\{ \|f - g\|_{C_B} + \delta \|g\|_{C_B^2} \right\}, \quad (2.5)$$

where

$$C_B^2[0, \infty) = \left\{ g \in C_B[0, \infty) : g', g'' \in C_B[0, \infty) \right\},$$

and the norm  $\|g\|_{C_B^2} = \|g\|_{C_B^2} + \|g'\|_{C_B} + \|g''\|_{C_B}$ . The following inequality

$$K(f; \delta) \leq M \left\{ \omega_2(f; \sqrt{\delta}) + \min(1, \delta) \|f\|_{C_B} \right\}, \quad (2.6)$$

holds for all  $\delta > 0$  and the constant  $M$  is independent of  $f$  and  $\delta$ .

**Theorem 3.** *If  $f \in C_B^2[0, \infty)$ , then we have*

$$|L_n^*(f; x) - f(x)| \leq \xi \|f\|_{C_B^2},$$

where

$$\xi := \xi_n(x) = \left\{ \frac{g'(1)}{g(1)} \frac{b_n}{n} + \frac{b_n}{n} + \frac{b_n^2}{n^2} \left( \frac{2g'(1)}{g(1)} + \frac{g''(1)}{g(1)} + 1 \right) \right\}.$$

*Proof.* From the Taylor expansion of  $f$ , the linearity of the operators  $L_n^*$  and (2.1), we have

$$L_n^*(f; x) - f(x) = f'(x) L_n^*(s - x; x) + \frac{1}{2} f''(\eta) L_n^*((s - x)^2; x), \eta \in (x, s). \quad (2.7)$$

Since

$$L_n^*(s - x; x) = \frac{g'(1)}{g(1)} \frac{b_n}{n} + \frac{b_n}{n} \geq 0$$

for  $s \geq x$ , by considering Lemma and (2.7), we can write

$$\begin{aligned} |L_n^*(f; x) - f(x)| &\leq \left\{ \frac{g'(1)}{g(1)} \frac{b_n}{n} + \frac{b_n}{n} \right\} \|f'\|_{C_B} + \left\{ \frac{b_n^2}{n^2} \left( \frac{2g'(1)}{g(1)} + \frac{g''(1)}{g(1)} + 1 \right) \right\} \|f''\|_{C_B} \\ &\leq \left\{ \frac{g'(1)}{g(1)} \frac{b_n}{n} + \frac{b_n}{n} + \frac{b_n^2}{n^2} \left( \frac{2g'(1)}{g(1)} + \frac{g''(1)}{g(1)} + 1 \right) \right\} \|f\|_{C_B^2}. \end{aligned}$$

which completes the proof.  $\square$

**Theorem 4.** *Let  $f \in C_B[0, \infty)$ , then*

$$|L_n^*(f; x) - f(x)| \leq 2M \left\{ \omega_2(f; \sqrt{\delta}) + \min(1, \delta) \|f\|_{C_B} \right\},$$

where

$$\delta := \delta_n(x) = \frac{1}{2} \varsigma_n(x)$$

and  $M > 0$  is a constant which is independent of the functions  $f$  and  $\delta$ . Also,  $\varsigma_n(x)$  is the same as in the Theorem 3.

*Proof.* Suppose that  $g \in C_B^2[0, \infty)$ . From Theorem 3, we can write

$$\begin{aligned} |L_n^*(f; x) - f(x)| &\leq |L_n^*(f - g; x)| + |L_n^*(g; x) - g(x)| + |g(x) - f(x)| \quad (2.8) \\ &\leq 2 \|f - g\|_{C_B} + \varsigma \|g\|_{C_B^2} \\ &= 2 \left[ \|f - g\|_{C_B} + \delta \|g\|_{C_B^2} \right] \end{aligned}$$

The left-hand side of inequality (2.8) does not depend on the function  $g \in C_B^2[0, \infty)$ , so

$$|L_n^*(f; x) - f(x)| \leq 2K(f, \delta)$$

holds where  $K(f; \delta)$  is Peetre's  $K$ -functional defined by (2.5). By the relation between Peetre's  $K$  functional and the second modulus of smoothness given by (2.6), inequality (2.8) becomes

$$|L_n^*(f; x) - f(x)| \leq 2M \left\{ \omega_2(f; \sqrt{\delta}) + \min(1, \delta) \|f\|_{C_B} \right\}$$

hence we have the result.  $\square$

Now let us consider the Lipschitz type space with two parameters (see [13]).

$$Lip_M^{(\alpha_1, \alpha_2)}(\alpha) := \left\{ f \in C_B[0, \infty) : |f(t) - f(x)| \leq M \frac{|t - x|^\alpha}{(t + \alpha_1 x^2 + \alpha_2 x)^{\frac{\alpha}{2}}}; x, t \in [0, \infty) \right\}$$

for  $\alpha_1, \alpha_2 > 0$ ,  $M$  is a positive constant and  $\alpha \in [0, 1)$ .

**Theorem 5.** *Let  $f \in Lip_M^{(\alpha_1, \alpha_2)}(\alpha)$ . For all  $x > 0$ , we have*

$$|L_n^*(f; x) - f(x)| \leq M \left( \frac{\frac{b_n^2}{n^2} \left( \frac{2g'(1)}{g(1)} + \frac{g''(1)}{g(1)} + 1 \right)}{\alpha_1 x^2 + \alpha_2 x} \right)^{\frac{\alpha}{2}}$$

*Proof.* Let  $\alpha = 1$ .

$$\begin{aligned} |L_n^{\alpha,*}(f; x) - f(x)| &\leq \frac{e^{-nx}}{g(1)} \frac{n}{b_n} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} |f(t) - f(x)| dt \\ &\leq M \frac{e^{-nx}}{g(1)} \frac{n}{b_n} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} |t - x| dt \\ &\leq \frac{M}{\sqrt{\alpha_1 x^2 + \alpha_2 x}} L_n^{\alpha,*}(|t - x|; x) \\ &\leq M \left( \sqrt{\frac{L_n^{\alpha,*}((t - x)^2; x)}{\alpha_1 x^2 + \alpha_2 x}} \right) \\ &\leq M \frac{b_n}{n} \sqrt{\frac{\frac{2g'(1)}{g(1)} + \frac{g''(1)}{g(1)} + 1}{\alpha_1 x^2 + \alpha_2 x}}. \end{aligned}$$

Let  $\alpha \in (0, 1)$ . By applying Hölder inequality with  $p = \frac{1}{\alpha}$  and  $q = \frac{1}{1-\alpha}$

$$\begin{aligned}
|L_n^{\alpha,*}(f; x) - f(x)| &\leq \frac{e^{-nx}}{g(1)} \frac{n}{b_n} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} |f(t) - f(x)| dt \\
&\leq \left\{ \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \left( \frac{n}{b_n} \int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} |f(t) - f(x)| dt \right)^{\frac{1}{\alpha}} \right\}^{\alpha} \\
&\leq \left\{ \frac{n}{b_n} \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} |f(t) - f(x)|^{\frac{1}{\alpha}} dt \right\}^{\alpha} \\
&\leq M \left\{ \frac{n}{b_n} \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} \frac{|t-x|}{\sqrt{t+\alpha_1x^2+\alpha_2x}} dt \right\}^{\alpha} \\
&\leq \frac{M}{(\alpha_1x^2+\alpha_2x)^{\frac{\alpha}{2}}} \left\{ \frac{n}{b_n} \frac{e^{-nx}}{g(1)} \sum_{k=0}^{\infty} p_k\left(\frac{n}{b_n}x\right) \int_{\frac{k}{n}b_n}^{\frac{k+1}{n}b_n} |t-x| dt \right\}^{\alpha} \\
&\leq \frac{M}{(\alpha_1x^2+\alpha_2x)^{\frac{\alpha}{2}}} (L_n^{\alpha,*}(|t-x|; x))^{\alpha} \\
&\leq M \left( \frac{L_n^{\alpha,*}((t-x)^2; x)}{\alpha_1x^2+\alpha_2x} \right)^{\frac{\alpha}{2}} = M \left( \frac{\frac{b_n^2}{n^2} \left( \frac{2g'(1)}{g(1)} + \frac{g''(1)}{g(1)} + 1 \right)}{\alpha_1x^2+\alpha_2x} \right)^{\frac{\alpha}{2}}
\end{aligned}$$

□

### 3. APPROXIMATION PROPERTIES IN WEIGHTED SPACES

Now we present the weighted spaces of functions that appear in the paper. With this purpose we firstly introduce the function

We give approximation properties of the operators  $L_n^*$  of the weighted spaces of continuous functions with exponential growth on  $R_0^+ = [0, \infty)$  with the help of the weighted Korovkin type theorem proved by Gadjiev in [4], [5]. Therefore we consider the following weighted spaces of functions which are defined on the  $R_0^+$ .

Let  $\rho(x)$  be the weight function and  $M_f$  be a positive constant, we define

$$B_{\rho}(R_0^+) = \{f \in E(R_0^+) : |f(x)| \leq M_f \rho(x)\},$$

$$C_{\rho}(R_0^+) = \{f \in B(R_0^+) : f \text{ is continuous}\},$$

$$C_{\rho}^k(R_0^+) = \left\{ f \in C(R_0^+) : \lim_{x \rightarrow \infty} \frac{f(x)}{\rho(x)} = K_f < \infty \right\}.$$

It is obvious that  $C_\rho^k(R_0^+) \subset C_\rho(R_0^+) \subset B_\rho(R_0^+)$ .  $B_\rho(R_0^+)$  is a linear normed space with the norm

$$\|f\|_\rho = \sup_{x \in R_0^+} \frac{|f(x)|}{\rho(x)}.$$

The following results on the sequence of positive linear operators in these spaces are given in [4], [5].

**Lemma 3.** *The sequence of positive linear operators  $(L_n)_{n \geq 1}$  which act from  $C_\rho(R_0^+)$  to  $B_\rho(R_0^+)$  if and only if there exists a positive constant  $k$  such that  $L_n(\rho; x) \leq k\rho(x)$ , i.e.  $\|L_n(\rho; x)\|_\rho \leq k$ .*

**Theorem 6.** *Let  $(L_n)_{n \geq 1}$  be the space of positive linear operators which act from  $C_\rho(R_0^+)$  to  $B_\rho(R_0^+)$  satisfying the conditions*

$$\lim_{n \rightarrow \infty} \|L_n(t^v; x) - x^v\|_\rho = 0, \quad v = 0, 1, 2,$$

*then for any function  $f \in C_\rho^k(R_0^+)$*

$$\lim_{n \rightarrow \infty} \|L_n f - f\|_\rho = 0.$$

**Lemma 4.** *Let  $\rho(x) = 1 + x^2$  be a weight function. If  $f \in C_\rho(R_0^+)$ , then*

$$\|L_n^*(\rho; x)\|_\rho \leq 1 + M.$$

*Proof.* Using (2.1) and (2.2), we have

$$\begin{aligned} L_n^*(\rho; x) &= 1 + x^2 + \frac{b_n}{n} x \left( \frac{g(1) + 2g'(1)}{g(1)} + 1 \right) + \frac{2b_n^2}{n^2} \frac{g'(1)}{g(1)} + \frac{b_n^2}{n^2} \frac{g''(1)}{g(1)} + \frac{b_n^2}{3n^2} \\ \|L_n^*(\rho; x)\|_\rho &= \sup_{x \in R_0^+} \frac{1}{1 + x^2} \left[ 1 + x^2 + \frac{b_n}{n} x \left( \frac{g(1) + 2g'(1)}{g(1)} + 1 \right) + \frac{2b_n^2}{n^2} \frac{g'(1)}{g(1)} + \frac{b_n^2}{n^2} \frac{g''(1)}{g(1)} + \frac{b_n^2}{3n^2} \right] \\ &\leq 1 + \frac{b_n}{n} \left( \frac{g(1) + 2g'(1)}{g(1)} + 1 \right) + \frac{2b_n^2}{n^2} \frac{g'(1)}{g(1)} + \frac{b_n^2}{n^2} \frac{g''(1)}{g(1)} + \frac{b_n^2}{3n^2} \end{aligned}$$

since  $\lim_{n \rightarrow \infty} \frac{b_n}{n} = 0$ , there exists a positive constant  $M$  such that

$$\|L_n^*(\rho; x)\|_\rho \leq 1 + M.$$

□

By using Lemma 3, we can easily see that the operators  $L_n^*$  defined by (1.4) act from  $C_\rho(R_0^+)$  to  $B_\rho(R_0^+)$ .

**Theorem 7.** *Let  $L_n^*$  be the sequence of linear positive operators defined by (1.4) and  $\rho(x) = 1 + x^2$ , then for each  $f \in C_\rho^k(R_0^+)$*

$$\lim_{n \rightarrow \infty} \|L_n^*(\rho; x) - f(x)\|_\rho = 0.$$

*Proof.* It is enough to show that the conditions of the weighted Korovkin type theorem given by Theorem 6. From (2.1), we can write

$$\lim_{n \rightarrow \infty} \|L_n^*(1; x) - 1\|_\rho = 0. \quad (3.1)$$



Using (2.2), we have

$$\|L_n^*(e_1; x) - e_1(x)\|_\rho = \frac{b_n}{n} \frac{g'(1)}{g(1)} + \frac{b_n}{n}$$

this implies that

$$\lim_{n \rightarrow \infty} \|L_n^*(e_1; x) - e_1(x)\|_\rho = 0. \quad (3.2)$$

From (2.3),

$$\begin{aligned} \|L_n^*(e_2; x) - e_2(x)\|_\rho &= \sup_{x \in R_0^+} \frac{1}{1+x^2} \left\{ \frac{b_n}{n} x \left( \frac{g(1)+2g'(1)}{g(1)} + 1 \right) + \frac{b_n^2}{n^2} \left( \frac{2g'(1)}{g(1)} + \frac{g''(1)}{g(1)} + \frac{1}{3} \right) \right\} \\ &\leq \frac{b_n}{n} \left( \frac{g(1)+2g'(1)}{g(1)} + 1 \right) + \frac{b_n^2}{n^2} \left( \frac{2g'(1)}{g(1)} + \frac{g''(1)}{g(1)} + \frac{1}{3} \right). \end{aligned}$$

Using the conditions (1.5), it follows that

$$\lim_{n \rightarrow \infty} \|L_n^*(e_2; x) - e_2(x)\|_\rho = 0. \quad (3.3)$$

From (3.1), (3.2) and (3.3) for  $v = 0, 1, 2$ , we have

$$\lim_{n \rightarrow \infty} \|L_n^*(e_v; x) - e_v(x)\|_\rho = 0.$$

If we apply Theorem 6, we obtain desired result.  $\square$

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